Mathematical Statistics (Math 46)
Take-Home Final Exam - Hartlaub
Due on Thursday, May 13, 1999

Solve all of the problems below. The point values for each problem are specified in parentheses. You may use resources in the library and elsewhere, but don't forget to note appropriate references. You may not discuss these problems with any other individuals, including faculty members and students.

1.(15) Making use of the fact that the chi-square distribution can be approximated with a normal distribution when ν, the number of degrees of freedom, is large, show that for large samples from normal populations

$$s^2 \geq \sigma_0^2\left[1 + z_\alpha\sqrt{\frac{2}{n-1}}\right]$$

is an approximate critical region of size α for testing the null hypothesis $\sigma^2 = \sigma_0^2$ against the alternative $\sigma^2 > \sigma_0^2$. Also construct the approximate critical region for testing this null hypothesis against the alternative $\sigma^2 < \sigma_0^2$.

2.(15) Let $\overline{X}$ denote the mean of a random sample of size 25 from a gamma distribution with $\alpha = 4$ and $\beta > 0$. Use the Central Limit Theorem to find an approximate 95.44% confidence interval for μ, the mean of a gamma distribution. [Hint: Base the confidence interval on the random variable

$$\frac{\overline{X} - 4\beta}{\sqrt{\frac{4\beta^2}{25}}} = \frac{5\overline{X}}{2\beta} - 10.]$$

3. Given a random sample of size n from a population having the p.d.f.

$$f(x;\theta) = \begin{cases} (\theta+1)x^\theta, & 0<x<1 \\ 0 & ,\textit{otherwise}. \end{cases}$$

find an estimator of θ

(10) a. using the method of moments.
(10) b. using the method of maximum likelihood.

4. Let $X_1, X_2, ..., X_n$ constitute a random sample from a $N(0, \sigma^2)$ population.

(10) a. Find the m.l.e. of σ^2.
(5) b. Find the m.l.e. of $1/\sigma^2$.
(10) c. Show that the m.l.e. of σ^2 in part (a) is an unbiased estimator of σ^2.

5.(15) Show that if $\hat{\theta}$ is a biased estimator of θ, then

$$E\left[(\hat{\theta}-\theta)^2\right] = Var(\hat{\theta}) + [b(\theta)]^2.$$

6.(10) Suppose the random variable X has a gamma distribution with mean $\mu_X = \alpha\beta$. Find $E[1/X]$.

7.(20) Suppose that X_1 and X_2 are independent standard normal random variables and that $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Find the joint distribution of Y_1 and Y_2 using one of the transformation of variables techniques. Do you recognize this distribution?

8. Consider a random sample from an exponential distribution, $X_i \sim Exp(\theta)$, $i=1,...,n$.

(10) a. Find the Cramér-Rao lower bound for the variance of all unbiased estimators of θ.
(5) b. Find the joint distribution of $X_1, X_2, ..., X_n$.
(10) c. Is the sample average a sufficient estimator for the parameter θ. Justify your answer.

(10) d. Use the m.g.f. technique to identify the distribution of $Y = \sum_{i=1}^{n} X_i$.

(10) e. Use the definition of sufficiency to prove that Y is sufficient estimator of θ.

9. Consider a random sample of size n from a population having the p.d.f.

$$f(x;\theta) = \begin{cases} \frac{2(\theta-x)}{\theta^2}, & 0<x<\theta; \\ 0 & , otherwise. \end{cases}$$

(10) a. Is $3\overline{X}$ an unbiased estimator of θ? (Justify your answer.)
(10) b. Is $3\overline{X}$ a consistent estimator of θ? (Justify your answer.)

10. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables, each with the uniform distribution on (0,1).

(5) a. Find $E[\overline{X}]$ and $Var[\overline{X}]$.
(10) b. What does Chebychev's inequality say about the probability that $\overline{X}$ is not between 0.4 and 0.6?
(5) c. How large must n be in order for Chebychev's inequality to guarantee that this probability is less than 0.01?

11. Explain how to simulate data from the c.d.f. $F(x) = 1 - \exp(-x/37)$ if you are given n Uniform(0, 1) random variables. (10)

12. A sample of 15 cigarettes of a certain brand was tested for nicotine content. The average content of these 15 cigarettes was found to be 20.3 mg and the standard deviation, s=3.0 mg.

(10) a. Construct a 95% confidence interval for the mean content for all cigarettes of this brand. (Assume that the random sample is from a normal population.)

(5) b. Interpret the interval you found in part (a).

13. Given a random sample of size n from a population with c.d.f. $F(x) = x^{\beta}$ for $0<x<1$, where $\beta>0$, obtain an estimator for β

(10) a. using the method of moments.

(10) b. using the method of maximum likelihood.

14. Suppose the mean weight of a rancher's cattle (live weight at the time of slaughter) has been 400 lb. A feed salesman has induced him to try a new supplement on 60 cattle. The mean weight of these cattle is 405 lb. with s.d. 20 lb. Is there evidence that the supplement is effective? (15)

15. If $\Psi_X(t) = \exp(3t+8t^2)$ is the moment generating function of the random variable X, find $P(-1 < X < 9)$. (10)

16. Consider the function

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{(\alpha-1)} e^{(-\frac{x}{\beta})}, & 0<x<\infty, \alpha>0, \beta>0 \\ 0 & , elsewhere \end{cases}$$

a. Verify that f(x) is a p.d.f. (5)

b. Suppose that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is a random sample from the p.d.f. given above with $\alpha=2$. Find the maximum likelihood estimator of β. (10)

17. A certain type of unit has an exponential time to failure, with mean two hours. Find the probability that at least one of a pair of units operating independently lasts four hours. (15)